The Markov basis of $K_{3,N}$

Johannes Rauh
MPI MIS
Inselstraße 22
04103 Leipzig
jarauh@gmx.net

Seth Sullivant
Department of Mathematics, NCSU
Box 8205,
Raleigh, NC 27695
smsulli2@ncsu.edu

June 27, 2014

This document explains how to obtain a Markov basis of the graphical model of the complete bipartite graph $K_{3,N}$ with binary nodes. The computations illustrate the theory developed in [6] that explains how to compute Markov bases of toric fiber products.

An HTML version of this document is available at `http://markov-bases.de/models/K3N/K3N`

Contents

1. Summary

We compute Markov bases of the binary (i.e. $d_i = 2$) hierarchical model of the complete bipartite graph $K_{3,N}$.

Theorem 1. *For any N, the Markov degree of the binary hierarchical model of the complete bipartite graph $K_{3,N}$ is at most 12.*

The degree of the kernel Markov basis is at most 6, the degree of the PF Markov basis is 4 and the lifting defect is 2. Therefore, another bound on the degree of $K_{3,N}$ is $4+2N$.

The proof of this theorem will take up Sections 2 to 5 of this manuscript. The proof will also give an explicit description of a Markov basis of $K_{3,N}$. In Section 6 we compare our theoretical bounds with Markov bases that were obtained with `4ti2` [1]. Both bounds are not sharp for $N \leq 3$.

The proof relies heavily on the lifting machinery developped in [6]. All the notation and all the notions are explained in detail in that manuscript.

The idea of the proof is to use the fact that $K_{3,N}$ is a toric fiber product of N copies of the three-star. The associated codimension zero product is a product of copies of the graph $\tilde{K}_4$ that arises from K_4 by filling the triangle $\{1, 2, 3\}$ filled (). The calculation is complicated by the fact that the marginal cone of is not normal. However, it turns out that all holes are vertices of the projected fibers. This allows to treat the holes in a systematic way by adding additional inequalities to the inequality description of the projected fibers. We compute the holes in Section 2.

The set of holes is described in Section 2. This allows to describe the projected fibers and to calculate a PF Markov basis (Section 3). The liftings are computed in Section 4, where it is also shown that the degree of the glued moves is at most 12. Section 5 presents a kernel Markov basis of degree six. As it turns out, all moves from the kernel Markov basis are redundant except for the quadratic moves. The appendix contains the kernel Markov basis (Appendix A.1), the PF Markov basis (Appendix A.2) and the lifts (Appendix A.3) in tensor notation.

The results in this manuscript were obtained with the help of `Normaliz` [2], `4ti2` [1] and `Macaulay2` [3]. This PDF-file contains attached files with code to reproduce the results. In PDF viewers that support attached files they can usually be accessed by right clicks on the symbol .

An HTML-version of the results, which may be better suited for reading on a screen, is available at `http://markov-bases.de/models/K3N/K3N.html`.

2. The holes of $\tilde{K}_4$

In this section we study the set of holes of $\mathbb{N}\tilde{K}_4$. The following lemma summarizes the most important properties:

Lemma 2. *The set of holes of $\mathbb{N}\mathcal{B}_{\tilde{K}_4}$ satisfies the following statements:*

1. *There are two fundamental holes h_1, h_2.*

2. *There is a partition $\mathcal{B}_{\tilde{K}_4} = \mathcal{B}_1 \dot\cup \mathcal{B}_2$ of the rows of $\mathcal{B}_{\tilde{K}_4}$ such that the set of holes is given by $(h_1 + \mathbb{N}\mathcal{B}_1) \dot\cup (h_2 + \mathbb{N}\mathcal{B}_2)$. The $\triangleright$-margins of $\mathcal{B}_i$ are linearly independent for $i = 1, 2$.*

3. *There are linear functionals $l_1, l_2 : \mathbb{Z}^{\mathcal{B}_{\tilde{K}_4}} \to \mathbb{Z}$ that satisfy the following: If a fiber $\mathbf{F}(b)$ has a hole $h \in (h_i + \mathbb{N}\mathcal{B}_i)$, then $l_{3-i}(v) > 0 = l_{3-i}(h)$.*

The lemma follows from the observations made in the remainder of this section. A Macaulay2-program that does the calculations in this section can be found here.

A computation with `Normaliz` (input/ output) shows that the Hilbert basis of the saturation of $\mathbb{N}\mathcal{B}_{\tilde{K}_4}$ contains two additional vectors h_1, h_2. Both restrict to the all-ones hole h on the K_4-marginals (that is, on all pair marginals). On the triangle-marginal, h_1 corresponds to XOR and h_2 corresponds to the opposite of XOR. They are the only fundamental holes, in the nomenclature of [4], as the following computations show:

$$h_1 + h_2 = \mathcal{B}_{\triangleright} \begin{bmatrix} 0000 \\ 0011 \\ 0101 \\ 0110 \\ 1000 \\ 1011 \\ 1101 \\ 1110 \end{bmatrix}, \qquad 2h_1 = \mathcal{B}_{\triangleright} \begin{bmatrix} 0000 \\ 0001 \\ 0110 \\ 0111 \\ 1010 \\ 1011 \\ 1100 \\ 1101 \end{bmatrix}, \qquad 2h_2 = \mathcal{B}_{\triangleright} \begin{bmatrix} 0010 \\ 0011 \\ 0100 \\ 0101 \\ 1000 \\ 1001 \\ 1110 \\ 1111 \end{bmatrix}.$$

Consider one of the fundamental holes h_i. According to [4], we need to do the following:

1. Find the minimal non-negative solutions (λ, μ) of $h_i + \mathcal{B}\lambda = \mathcal{B}\mu$. This can be done, for example, using `4ti2`'s command `zsolve`.

   Input: `h1.mat`/ `h2.mat`, `h1.rhs`/ `h2.rhs`,
   `h1.sign`/ `h2.sign`.

   Output: `h1.zinhom`/ `h2.zinhom`.

2. Drop the μ's and interprete the λ's (first half of the matrix) as the exponent vectors of monomials that generate a monomial ideal I.

3. Compute the standard monomials. In Macaulay2, this can be done using the command `standardPairs`.

The result is the following: Let

$$\mathcal{B}_1 = (b_{0000}, b_{1100}, b_{1010}, b_{0110}, b_{0001}, b_{1101}, b_{1011}, b_{0111}),$$
$$\mathcal{B}_2 = (b_{1000}, b_{0100}, b_{0010}, b_{1110}, b_{1001}, b_{0101}, b_{0011}, b_{1111})$$

($\mathcal{B}_1$ corresponds to XOR on the first three nodes, and $\mathcal{B}_2$ corresponds to its opposite). The holes derived from h_1 are of the form $h_1 + \mathcal{B}_1\lambda$, where $\lambda \in \mathbb{N}^8$. By symmetry, the holes derived from h_2 are $h_2 + \mathcal{B}_2\mu$, where $\mu \in \mathbb{N}^8$.

Consider the two linear forms

$$
\begin{aligned}
l_1 &= y^{123}_{000} + y^{123}_{011} + y^{123}_{101} + y^{123}_{110},\\
l_2 &= y^{123}_{001} + y^{123}_{010} + y^{123}_{100} + y^{123}_{111},
\end{aligned}
$$

where y^{123}_{ijk} counts the coordinates where the (123)-marginal is equal to ijk (that is, l_1 counts the XOR-part of the triangle-margin, and l_2 counts the opposite XOR-part). Then

$$
l_1(h_1 + \mathcal{B}_1\lambda) > 0, \quad l_2(h_1 + \mathcal{B}_1\lambda) = 0, \quad l_1(h_2 + \mathcal{B}_2\mu) = 0, \quad l_2(h_2 + \mathcal{B}_2\mu) > 0.
$$

Therefore, each hole is either derived from h_1 or from h_2, that is, $(h_1+\mathbb{N}\mathcal{B}_1)\cap(h_2+\mathbb{N}\mathcal{B}_2) = \emptyset$.

Each set $\mathcal{B}_i$ is linearly independent. Hence different choices of the λ (or μ) give different holes. Moreover, also the vectors $\gtrdot$-marginals of $\mathcal{B}_i$ are linearly independent for each i. Therefore, no two holes of the same type have the same $\gtrdot$-marginals. Therefore, no projected fiber contains more than one hole of the same type. A projected fiber can have one hole of each type, though (for example, the holes $h_1 + \mathcal{B}_1(1,\dots,1)^t$ and $h_2 + \mathcal{B}_2(1,\dots,1)^t$ have the same pair marginals and lie in the same fiber).

Let $h = h_1 + \mathcal{B}_1\lambda$ be a hole. If v belongs to the same projected fiber as h, then $v = \mathcal{B}_1\lambda' + \mathcal{B}_2\mu'$ with $\lambda', \mu' \in \mathbb{N}^8$. As mentioned above, the $\gtrdot$-pair marginals of $\mathcal{B}_1$ are linearly independent. Therefore, if $\mu' = 0$, then, since h and v have the same $\gtrdot$-marginals, $h = v$. Hence, if $v \neq h$ is not a hole itself, $\mu' \neq 0$. It follows that $l_2(m) > 0 = l_2(h)$.

3. The projected fiber Markov basis

By Lemma 2, every hole is a vertex of its projected fiber, supported either by l_1 or l_2. Thus we can do the following: We start with an inequality description of the semigroup $\mathbb{N}\tilde{K}_4$. This gives us a set of valid inequalities $Du \geq c$ for each projected fiber. These inequalities are also valid for the holes. We augment the matrix D by two additional rows corresponding to l_1 and l_2 and denote the augmented matrix by D'. Then each projected fiber equals a solution set of linear inequalities of the form $D'u \geq c'$. Therefore, any inequality Markov basis of D' can be used as a PF Markov basis.

Each projected fiber is a subset of $\mathbb{Z}^8$, with basis $e_{000}, e_{001}, \dots, e_{111}$. Let $y_{000}, \dots, y_{111}$ be the corresponding coordinates. In a projected fiber, there are relations

$$
\begin{aligned}
y_{011} &= y^1_0 - y_{000} - y_{001} - y_{010},\\
y_{101} &= y^2_0 - y_{000} - y_{001} - y_{100},\\
y_{110} &= y^3_0 - y_{000} - y_{010} - y_{100},\\
y_{111} &= 1 - y^1_0 - y^2_0 - y^3_0 + 2y_{000} + y_{001} + y_{010} + y_{100},
\end{aligned}
$$

where y^i_j is the sum of those marginals where the ith entry equals j. Since the projected fiber is four-dimensional, these are all relations. An independent set of coordinates is

given by $y_{000}, y_{001}, y_{010}, y_{100}$ (that is, those coordinates with at most one one). According to `Normaliz` (input/ output), they satisfy inequalities of the form

$$\begin{gathered} ? \le y_{000}, \quad ? \le y_{001}, \quad ? \le y_{010}, \quad ? \le y_{100}, \\ ? \le y_{000} + y_{001} \le ?, \quad ? \le y_{000} + y_{010} \le ?, \quad ? \le y_{000} + y_{100} \le ?, \\ y_{000} + y_{001} + y_{010} \le ?, \quad y_{000} + y_{001} + y_{100} \le ?, \quad y_{000} + y_{010} + y_{100} \le ?, \\ ? \le 2y_{000} + y_{001} + y_{010} + y_{100}. \end{gathered}$$

To get rid of the holes, we need to add the inequalities with linear parts l_1, l_2. In coordinates, they take the form

$$? \le 2(y_{000} + y_{001} + y_{010} + y_{100}) \le ?.$$

The columns of the corresponding matrix D spans a lattice, and `4ti2` computes its Markov basis (input/ output). Since the first four rows of D form a unit matrix, D is easy to invert: The first four coordinates of the Markov basis of D give the PF Markov basis in the coordinates $y_{000}, y_{001}, y_{010}, y_{100}$. In tableau notation, PF Markov basis consists of the 16 moves, that are (up to symmetry; that is, up to a permutation of the columns) of the form

$$\begin{bmatrix} 0 & 0 & a \\ 1 & 1 & b \end{bmatrix} - \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & b \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

4. Lifting the PF Markov basis

In this section we compute the lifts. We use the algorithm described in [6] to compute the lifting as an inequality Markov basis. Analyzing the results we find that the maximal degree of a glue of lifts is bounded by 12. The file lift.m2 contains Macaulay2 code that does the calculations in this section. It makes use of further routines from M2routines.m2.

First consider

$$g = \begin{bmatrix} 0 & 0 & a \\ 1 & 1 & b \end{bmatrix} - \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & b \end{bmatrix}.$$

If $b = a$, then g lifts to

$$\begin{bmatrix} 0 & 0 & a & c \\ 1 & 1 & a & c \end{bmatrix} - \begin{bmatrix} 0 & 1 & a & c \\ 1 & 0 & a & c \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & a & c \\ 1 & 1 & a & \overline{c} \\ 1 & d & \overline{a} & c \\ 0 & d & \overline{a} & \overline{c} \end{bmatrix} - \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & a & \overline{c} \\ 0 & d & \overline{a} & c \\ 1 & d & \overline{a} & \overline{c} \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 0 & 0 & a & c \\ 1 & 1 & a & \overline{c} \\ d & 1 & \overline{a} & c \\ d & 0 & \overline{a} & \overline{c} \end{bmatrix} - \begin{bmatrix} 0 & 1 & a & c \\ 1 & 0 & a & \overline{c} \\ d & 0 & \overline{a} & c \\ d & 1 & \overline{a} & \overline{c} \end{bmatrix}.$$

The lifting defect is at most two. Any move $\tilde{m}$ that arises by gluing lifts of g satisfies

$$\xi(\tilde{m}^+) - g^+ \leq \begin{bmatrix} 1 & 0 & \overline{a} \\ 0 & 0 & \overline{a} \\ 1 & 1 & \overline{a} \\ 0 & 1 & \overline{a} \end{bmatrix}.$$

Therefore, $\deg(\tilde{m}) \leq 6$.

If $b = \overline{a}$, then g lifts to

$$\begin{bmatrix} 0 & 0 & a & c \\ 1 & 1 & \overline{a} & c \end{bmatrix} - \begin{bmatrix} 0 & 1 & a & c \\ 1 & 0 & \overline{a} & c \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & a & c \\ 1 & 1 & \overline{a} & \overline{c} \\ 0 & 0 & \overline{a} & \overline{c} \\ 0 & 1 & \overline{a} & c \end{bmatrix} - \begin{bmatrix} 0 & 1 & a & c \\ 1 & 0 & \overline{a} & \overline{c} \\ 0 & 1 & \overline{a} & \overline{c} \\ 0 & 0 & \overline{a} & c \end{bmatrix},$$

$$\text{and} \quad \begin{bmatrix} 0 & 0 & a & c \\ 1 & 1 & \overline{a} & \overline{c} \\ 1 & 0 & a & \overline{c} \\ 1 & 1 & a & c \end{bmatrix} - \begin{bmatrix} 0 & 1 & a & c \\ 1 & 0 & \overline{a} & \overline{c} \\ 1 & 1 & a & \overline{c} \\ 1 & 0 & a & c \end{bmatrix}.$$

The lifting defect is at most two. Any move $\tilde{m}$ that arises by gluing lifts of g satisfies

$$\xi(\tilde{m}^+) - g^+ \leq \begin{bmatrix} 0 & 0 & \overline{a} \\ 0 & 1 & \overline{a} \\ 1 & 0 & a \\ 1 & 1 & a \end{bmatrix}.$$

Therefore, $\deg(\tilde{m}) \leq 6$.

For

$$g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

the lifts are (up to symmetry, exchanging the first two columns and state switching)

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 1 & b \\ 1 & 1 & 0 & a \\ 1 & 1 & 1 & b \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 0 & a \\ 1 & 0 & 1 & b \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 1 & b \\ 1 & 1 & 0 & b \\ 1 & 1 & 1 & a \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 0 & b \\ 1 & 0 & 1 & a \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 1 & \overline{a} \\ 1 & 1 & 0 & a \\ 1 & 1 & 1 & a \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & \overline{a} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & a \\ 1 & 0 & 1 & \overline{a} \\ 0 & 1 & 0 & a \\ 0 & 1 & 1 & a \\ 1 & 0 & 0 & a \\ 0 & 0 & 0 & \overline{a} \end{bmatrix}.$$

The lifting defect is at most two. Any move $\tilde{m}$ that arises by gluing lifts of g satisfies

$$\xi(\tilde{m}^+) - g^+ \leq \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore, $\deg(\tilde{m}) \leq 12$.

For

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

the lifts are (up to symmetry, exchanging the first two columns and state switching)

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & a \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & a \\ 1 & 1 & 1 & b \\ 0 & 0 & 1 & b \\ 0 & 1 & 0 & a \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & a \\ 1 & 1 & 1 & b \\ 0 & 0 & 1 & a \\ 0 & 1 & 0 & b \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & 1 & \overline{a} \\ 1 & 0 & 1 & a \\ 1 & 1 & 0 & a \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & \overline{a} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & a \\ 1 & 1 & 1 & \overline{a} \\ 0 & 0 & 1 & b \\ 0 & 1 & 0 & a \\ 1 & 0 & 0 & a \\ 0 & 0 & 0 & \overline{a} \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & \overline{a} \\ 0 & 1 & 1 & a \\ 1 & 0 & 1 & a \\ 1 & 1 & 0 & a \\ 0 & 0 & 0 & \overline{a} \\ 1 & 0 & 0 & \overline{a} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \overline{a} \\ 1 & 1 & 1 & a \\ 0 & 0 & 1 & a \\ 0 & 1 & 0 & a \\ 1 & 0 & 0 & \overline{a} \\ 0 & 0 & 0 & \overline{a} \end{bmatrix}$$

The lifting defect is at most two. Any move $\tilde{m}$ that arises by gluing lifts of g satisfies

$$\xi(\tilde{m}^+) - g^+ \leq \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore, $\deg(\tilde{m}) \leq 12$.

5. The kernel Markov basis in tableau notation

The Markov basis of $\triangleright$ computed by `4ti2` has 20 elements of degrees four and six (input/ output). In tableau notation, it consists of the following moves (up to symmetry, involving permuting the first three columns):

$$\begin{bmatrix} a000 \\ b011 \\ a101 \\ b110 \end{bmatrix} - \begin{bmatrix} a001 \\ b010 \\ a100 \\ b111 \end{bmatrix}, \quad \begin{bmatrix} abc0 \\ abc0 \\ \overline{a}\overline{b}\overline{c}0 \\ \overline{a}bc1 \\ a\overline{b}c1 \\ ab\overline{c}1 \end{bmatrix} - \begin{bmatrix} abc1 \\ abc1 \\ \overline{a}\overline{b}\overline{c}1 \\ \overline{a}bc0 \\ a\overline{b}c0 \\ ab\overline{c}0 \end{bmatrix}.$$

Recall that the kernel Markov basis consists of quadratic moves of the form

$$\begin{bmatrix} abcD\ E \\ abcD'E' \end{bmatrix} - \begin{bmatrix} abcD\ E' \\ abcD'E \end{bmatrix}$$

and lifts that can be constructed from the moves in the Markov basis of $\triangleright$ by adding constant columns in the tableau notation. The lifted moves have degree 4 and 6, and so the kernel Markov basis has degree 6.

Let us show that all lifted moves from the kernel Markov basis are redundant; that is: We only need the quadratic moves from the kernel Markov basis. In fact, consider, for example, a lift m of degree six. The glues of lifts of the PF Markov basis contain the quadratic moves of the form

$$\begin{bmatrix} a\ b\ c\ d\ E \\ a'b'c'd'E \end{bmatrix} - \begin{bmatrix} a'b\ c\ d\ E \\ a\ b'c'd'E \end{bmatrix},$$

and so on. Applying such quadratic moves reduces m to zero:

$$\begin{bmatrix} abc0E \\ abc0E \\ \overline{a}\overline{b}\overline{c}0E \\ \overline{a}bc1E \\ a\overline{b}c1E \\ ab\overline{c}1E \end{bmatrix} \to \begin{bmatrix} \overline{a}bc0E \\ abc0E \\ a\overline{b}\overline{c}0E \\ \overline{a}bc1E \\ a\overline{b}c1E \\ ab\overline{c}1E \end{bmatrix} \to \begin{bmatrix} \overline{a}bc0E \\ a\overline{b}c0E \\ ab\overline{c}0E \\ \overline{a}bc1E \\ a\overline{b}c1E \\ ab\overline{c}1E \end{bmatrix} \to \begin{bmatrix} \overline{a}bc0E \\ a\overline{b}c0E \\ ab\overline{c}0E \\ abc1E \\ a\overline{b}c1E \\ \overline{a}b\overline{c}1E \end{bmatrix} \to \begin{bmatrix} \overline{a}bc0E \\ a\overline{b}c0E \\ ab\overline{c}0E \\ abc1E \\ abc1E \\ \overline{a}\overline{b}\overline{c}1E \end{bmatrix} = \begin{bmatrix} abc1E \\ abc1E \\ \overline{a}\overline{b}\overline{c}1E \\ \overline{a}bc0E \\ a\overline{b}c0E \\ ab\overline{c}0E \end{bmatrix}.$$

The quartic moves of the kernel Markov basis can be reduced similarly.

6. Comparison with computational results

For $N \leq 3$, the Markov basis of $K_{3,N}$ can be computed (within reasonable time) using `4ti2`. The Markov degrees are:

N	1	2	3
deg	2	4	6

These three computed degrees are much smaller than the theoretical bound of $\min\{4+2N, 12\}$.

$K_{3,1}$ is a tree; hence the Markov degree is two. The additional moves obtained by lifting the PF Markov basis are not necessary. For a zero-fold toric fiber product it is no wonder that our bound is far from being tight.

A Markov basis of $K_{3,2}$ was computed in [5] by interpreting $K_{3,2}$ as a TFP of three two-stars. Here, we interprete these moves from the viewpoint of our above computations. The Markov basis of [5] consists of two kinds of quadrics and quartics. First, there are the codimension-zero quadrics. Second, the quadrics of the two three-stars glue together and yield further quadrics. Observe that the two interpretations of $K_{3,2}$ interchange the roles of the codimension-zero quadrics and the glued quadrics: The codimension-zero quadrics of $K_{1,2} \times_{\mathcal{A}'} K_{1,2} \times_{\mathcal{A}'} K_{1,2}$ correspond to the glued quadrics of $K_{3,1} \times_{\mathcal{A}} K_{3,1}$, and vice versa.

The quartics are of the form

$$Q = \begin{bmatrix} 0 & a_{00} & b_{00} & 0 & 0 \\ 1 & a_{01} & b_{01} & 0 & 1 \\ 1 & a_{10} & b_{10} & 1 & 0 \\ 0 & a_{11} & b_{11} & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & a_{00} & b_{00} & 0 & 0 \\ 0 & a_{01} & b_{01} & 0 & 1 \\ 0 & a_{10} & b_{10} & 1 & 0 \\ 1 & a_{11} & b_{11} & 1 & 1 \end{bmatrix}$$

for some $l_{ij}, m_{ij} \in \{0,1\}$, modulo a permutation of the first three rows. All of these quadrics are glues of, say, m and m'. If either $(a_{00}, b_{00}) = (a_{01}, b_{01})$ or $(a_{10}, b_{10}) = (a_{11}, b_{11})$, then m is a quadric; that is

$$m = \begin{bmatrix} 1 & a_{10} & b_{10} & 1 \\ 0 & a_{11} & b_{11} & 1 \end{bmatrix} - \begin{bmatrix} 0 & a_{10} & b_{10} & 1 \\ 1 & a_{11} & b_{11} & 1 \end{bmatrix}, \text{ or } m = \begin{bmatrix} 1 & a_{00} & b_{00} & 1 \\ 0 & a_{01} & b_{01} & 1 \end{bmatrix} - \begin{bmatrix} 0 & a_{00} & b_{00} & 1 \\ 1 & a_{01} & b_{01} & 1 \end{bmatrix}.$$

Otherwise, m is the quartic

$$m = \begin{bmatrix} 0 & a_{00} & b_{00} & 0 \\ 1 & a_{01} & b_{01} & 0 \\ 1 & a_{10} & b_{10} & 1 \\ 0 & a_{11} & b_{11} & 1 \end{bmatrix} - \begin{bmatrix} 1 & a_{00} & b_{00} & 0 \\ 0 & a_{01} & b_{01} & 0 \\ 0 & a_{10} & b_{10} & 1 \\ 1 & a_{11} & b_{11} & 1 \end{bmatrix}.$$

That is, m is a lift of

$$g = \begin{bmatrix} 0 & a_{00} & b_{00} \\ 1 & a_{01} & b_{01} \\ 1 & a_{10} & b_{10} \\ 0 & a_{11} & b_{11} \end{bmatrix} - \begin{bmatrix} 1 & a_{00} & b_{00} \\ 0 & a_{01} & b_{01} \\ 0 & a_{10} & b_{10} \\ 1 & a_{11} & b_{11} \end{bmatrix}.$$

A g of this form may be quadratic or quartic, depending on the (a_{ij}, b_{ij}). One can see that projecting the Markov basis of $K_{3,2}$ gives the full PF Markov basis. Therefore, the reason that the bound is not sharp is not that the PF Markov basis is too large, but that not all glues are necessary.

A. Intermediate results in tensor notation

In this appendix we present some of the above results in tensor notation. These results are straightforward translations of the output of `4ti2`, without factoring out any symmetry or other structures, and so they should be considered as intermediate steps on the way from the raw `4ti2`-output towards the form summarized above.

The tensor notation is as follows: Any state $x \in \{0,1\}^4$ corresponds to a binary string $x_4x_3x_2x_1$, which can be considered as the binary representation of a natural number. The three least-significant bits $x_3x_2x_1$ correspond to the first group of nodes in $K_{3,N}$. The states appear at the following positions:

$$\begin{array}{cc} 0000\,0001 & 0100\,0101 \\ 0010\,0011 & 0110\,0111 \\[4pt] 1000\,1001 & 1100\,1101 \\ 1010\,1011 & 1110\,1111 \end{array}.$$

Entries of modulus one are replaced by their signs, and negative numbers are indicated by a $\overline{\text{bar}}$.

A.1. The kernel Markov basis in tensor notation

The Markov basis of $\triangleright$ computed by `4ti2` has 20 elements of degrees four and six:

$$\begin{array}{cc} +0 & -0 \\ -0 & +0 \\[4pt] -0 & +0 \\ +0 & -0 \end{array},\quad
\begin{array}{cc} 0+ & 0- \\ 0- & 0+ \\[4pt] 0- & 0+ \\ 0+ & 0- \end{array},\quad
\begin{array}{cc} +- & -+ \\ 00 & 00 \\[4pt] -+ & +- \\ 00 & 00 \end{array},\quad
\begin{array}{cc} 00 & 00 \\ +- & -+ \\[4pt] 00 & 00 \\ -+ & +- \end{array},\quad
\begin{array}{cc} +- & 00 \\ -+ & 00 \\[4pt] -+ & 00 \\ +- & 00 \end{array},\quad
\begin{array}{cc} 00 & +- \\ 00 & -+ \\[4pt] 00 & -+ \\ 00 & +- \end{array}$$

$$\begin{array}{cc} +0 & 0- \\ -0 & 0+ \\[4pt] -0 & 0+ \\ +0 & 0- \end{array},\quad
\begin{array}{cc} 0+ & -0 \\ 0- & +0 \\[4pt] 0- & +0 \\ 0+ & -0 \end{array},\quad
\begin{array}{cc} +- & 00 \\ 00 & -+ \\[4pt] -+ & 00 \\ 00 & +- \end{array},\quad
\begin{array}{cc} 00 & -+ \\ +- & 00 \\[4pt] 00 & +- \\ -+ & 00 \end{array},\quad
\begin{array}{cc} +0 & -0 \\ 0- & 0+ \\[4pt] -0 & +0 \\ 0+ & 0- \end{array},\quad
\begin{array}{cc} 0+ & 0- \\ -0 & +0 \\[4pt] 0- & 0+ \\ +0 & -0 \end{array}$$

$$\begin{array}{cc} 2- & -0 \\ -0 & 0+ \\[4pt] \bar{2}+ & +0 \\ +0 & 0- \end{array},\quad
\begin{array}{cc} +\bar{2} & 0+ \\ 0+ & -0 \\[4pt] -2 & 0- \\ 0- & +0 \end{array},\quad
\begin{array}{cc} +0 & 0- \\ \bar{2}+ & +0 \\[4pt] -0 & 0+ \\ 2- & -0 \end{array},\quad
\begin{array}{cc} 0+ & -0 \\ +\bar{2} & 0+ \\[4pt] 0- & +0 \\ -2 & 0- \end{array}$$

$$\begin{array}{cc} +0 & \bar{2}+ \\ 0- & +0 \\[4pt] -0 & 2- \\ 0+ & -0 \end{array},\quad
\begin{array}{cc} 0+ & +\bar{2} \\ -0 & 0+ \\[4pt] 0- & -2 \\ +0 & 0- \end{array},\quad
\begin{array}{cc} 0+ & -0 \\ -0 & 2- \\[4pt] 0- & +0 \\ +0 & \bar{2}+ \end{array},\quad
\begin{array}{cc} +0 & 0- \\ 0- & -2 \\[4pt] -0 & 0+ \\ 0+ & +\bar{2} \end{array}.$$

A.2. The projected fiber Markov basis in tensor notation

The PF Markov basis consists of the 16 $2 \times 2 \times 2$-tableaus

$$\begin{matrix} +- & 00 \\ -+ & 00 \end{matrix}, \quad \begin{matrix} 00 & +- \\ 00 & -+ \end{matrix}, \quad \begin{matrix} +0 & -0 \\ -0 & +0 \end{matrix}, \quad \begin{matrix} 0+ & 0- \\ 0- & 0+ \end{matrix}, \quad \begin{matrix} +- & -+ \\ 00 & 00 \end{matrix}, \quad \begin{matrix} 00 & 00 \\ +- & -+ \end{matrix},$$

$$\begin{matrix} +0 & 0- \\ -0 & 0+ \end{matrix}, \quad \begin{matrix} 0+ & -0 \\ 0- & +0 \end{matrix}, \quad \begin{matrix} +- & 00 \\ 00 & -+ \end{matrix}, \quad \begin{matrix} 00 & -+ \\ +- & 00 \end{matrix}, \quad \begin{matrix} +0 & -0 \\ 0- & 0+ \end{matrix}, \quad \begin{matrix} 0- & 0+ \\ +0 & -0 \end{matrix},$$

$$\begin{matrix} +- & +- \\ -+ & -+ \end{matrix}, \qquad \begin{matrix} +- & -+ \\ +- & -+ \end{matrix}, \qquad \begin{matrix} ++ & -- \\ -- & ++ \end{matrix},$$

$$\begin{matrix} +- & -+ \\ -+ & +- \end{matrix}.$$

A.3. Lifting in tensor notation

For

$$\begin{matrix} +- & 00 \\ -+ & 00 \end{matrix}$$

there are ten lifts:

$$\begin{matrix} +- & 00 \\ -+ & 00 \\ 00 & 00 \\ 00 & 00 \end{matrix}, \qquad \begin{matrix} 00 & 00 \\ 00 & 00 \\ +- & 00 \\ -+ & 00 \end{matrix}$$

$$\begin{matrix} +- & 00 \\ 00 & -+ \\ 00 & 00 \\ -+ & +- \end{matrix}, \quad \begin{matrix} +- & -+ \\ 00 & 00 \\ 00 & +- \\ -+ & 00 \end{matrix}, \quad \begin{matrix} 00 & -+ \\ +- & 00 \\ -+ & +- \\ 00 & 00 \end{matrix}, \quad \begin{matrix} 00 & 00 \\ +- & -+ \\ -+ & 00 \\ 00 & +- \end{matrix},$$

$$\begin{matrix} 0+ & -0 \\ 0- & +0 \\ -0 & +0 \\ +0 & -0 \end{matrix}, \quad \begin{matrix} +0 & -0 \\ -0 & +0 \\ 0- & +0 \\ 0+ & -0 \end{matrix}, \quad \begin{matrix} 0+ & 0- \\ 0- & 0+ \\ -0 & 0+ \\ +0 & 0- \end{matrix}, \quad \begin{matrix} +0 & 0- \\ -0 & 0+ \\ 0- & 0+ \\ 0+ & 0- \end{matrix}.$$

For

$$\begin{matrix} +0 & 0- \\ -0 & 0+ \end{matrix}$$

there are six lifts:

$$\begin{array}{cc} +0 & 0- \\ -0 & 0+ \\ 00 & 00 \\ 00 & 00 \end{array}, \qquad \begin{array}{cc} 00 & 00 \\ 00 & 00 \\ +0 & 0- \\ -0 & 0+ \end{array}$$

$$\begin{array}{cc} 00 & +- \\ 00 & -+ \\ +0 & -0 \\ -0 & +0 \end{array}, \qquad \begin{array}{cc} 0+ & 0- \\ 0- & 0+ \\ +- & 00 \\ -+ & 00 \end{array}, \qquad \begin{array}{cc} +- & 00 \\ -+ & 00 \\ 0+ & 0- \\ 0- & 0+ \end{array}, \qquad \begin{array}{cc} +0 & -0 \\ -0 & +0 \\ 00 & +- \\ 00 & -+ \end{array}.$$

For

$$\begin{array}{cc} +- & +- \\ -+ & -+ \end{array}$$

there are 21 lifts:

$$\begin{array}{cc} 0+ & -0 \\ 0- & +0 \\ -0 & 0+ \\ +0 & 0- \end{array},$$

$$\begin{array}{cc} +- & +- \\ -+ & -+ \\ 00 & 00 \\ 00 & 00 \end{array}, \qquad \begin{array}{cc} 00 & 00 \\ 00 & 00 \\ +- & +- \\ -+ & -+ \end{array}, \qquad \begin{array}{cc} +- & 00 \\ -+ & 00 \\ 00 & +- \\ 00 & -+ \end{array}, \qquad \begin{array}{cc} 00 & +- \\ 00 & -+ \\ +- & 00 \\ -+ & 00 \end{array},$$

$$\begin{array}{cc} 2\overline{2} & 00 \\ -+ & -+ \\ -+ & +- \\ 00 & 00 \end{array}, \qquad \begin{array}{cc} +- & +- \\ \overline{2}2 & 00 \\ 00 & 00 \\ +- & -+ \end{array}, \qquad \begin{array}{cc} 2- & 0- \\ \overline{2}+ & 0+ \\ -0 & +0 \\ +0 & -0 \end{array}, \qquad \begin{array}{cc} +\overline{2} & +0 \\ -2 & -0 \\ 0+ & 0- \\ 0- & 0+ \end{array},$$

$$\begin{array}{cc} 00 & \overline{2}2 \\ +- & +- \\ -+ & +- \\ 00 & 00 \end{array}, \qquad \begin{array}{cc} +- & +- \\ 00 & \overline{2}2 \\ 00 & 00 \\ -+ & +- \end{array}, \qquad \begin{array}{cc} 0+ & \overline{2}+ \\ 0- & 2- \\ -0 & +0 \\ +0 & -0 \end{array}, \qquad \begin{array}{cc} +0 & +\overline{2} \\ -0 & -2 \\ 0- & 0+ \\ 0+ & 0- \end{array},$$

$$\begin{array}{cc} +- & -+ \\ 00 & 00 \\ \overline{2}2 & 00 \\ +- & +- \end{array}, \qquad \begin{array}{cc} 00 & 00 \\ +- & -+ \\ +- & +- \\ \overline{2}2 & 00 \end{array}, \qquad \begin{array}{cc} +0 & -0 \\ -0 & +0 \\ \overline{2}+ & 0+ \\ 2- & 0- \end{array}, \qquad \begin{array}{cc} 0+ & 0- \\ 0- & 0+ \\ +\overline{2} & +0 \\ -2 & -0 \end{array},$$

$$\begin{matrix}+- & -+\\ 00 & 00\\ 00 & 2\overline{2}\\ -+ & -+\end{matrix}, \qquad \begin{matrix}00 & 00\\ +- & -+\\ -+ & -+\\ 00 & 2\overline{2}\end{matrix}, \qquad \begin{matrix}+0 & -0\\ -0 & +0\\ 0- & 2-\\ 0+ & \overline{2}+\end{matrix}, \qquad \begin{matrix}0+ & 0-\\ 0- & 0+\\ -0 & -2\\ +0 & +\overline{2}\end{matrix}.$$

For

$$\begin{matrix}+- & -+\\ -+ & +-\end{matrix}$$

there are 40 lifts:

$$\begin{matrix}+- & -+\\ -+ & +-\\ 00 & 00\\ 00 & 00\end{matrix}, \qquad \begin{matrix}00 & 00\\ 00 & 00\\ +- & -+\\ -+ & +-\end{matrix}, \qquad \begin{matrix}+- & 00\\ -+ & 00\\ 00 & -+\\ 00 & +-\end{matrix}, \qquad \begin{matrix}00 & +-\\ 00 & -+\\ -+ & 00\\ +- & 00\end{matrix},$$

$$\begin{matrix}+- & -+\\ 00 & 00\\ 00 & 00\\ -+ & +-\end{matrix}, \qquad \begin{matrix}00 & 00\\ +- & -+\\ -+ & +-\\ 00 & 00\end{matrix}, \qquad \begin{matrix}+0 & -0\\ -0 & +0\\ 0- & 0+\\ 0+ & 0-\end{matrix}, \qquad \begin{matrix}0+ & 0-\\ 0- & 0+\\ -0 & +0\\ +0 & -0\end{matrix},$$

$$\begin{matrix}2\overline{2} & -+\\ -+ & 00\\ -+ & 00\\ 00 & +-\end{matrix}, \qquad \begin{matrix}+- & 00\\ \overline{2}2 & +-\\ 00 & -+\\ +- & 00\end{matrix}, \qquad \begin{matrix}2- & -0\\ \overline{2}+ & +0\\ -0 & 0+\\ +0 & 0-\end{matrix}, \qquad \begin{matrix}+\overline{2} & 0+\\ -2 & 0-\\ 0+ & -0\\ 0- & +0\end{matrix},$$

$$\begin{matrix}+- & \overline{2}2\\ 00 & +-\\ 00 & +-\\ -+ & 00\end{matrix}, \qquad \begin{matrix}00 & +-\\ +- & \overline{2}2\\ -+ & 00\\ 00 & +-\end{matrix}, \qquad \begin{matrix}+0 & \overline{2}+\\ -0 & 2-\\ 0- & +0\\ 0+ & -0\end{matrix}, \qquad \begin{matrix}0+ & +\overline{2}\\ 0- & -2\\ -0 & 0+\\ +0 & 0-\end{matrix},$$

$$\begin{matrix}+- & 00\\ 00 & -+\\ \overline{2}2 & +-\\ +- & 00\end{matrix}, \qquad \begin{matrix}00 & -+\\ +- & 00\\ +- & 00\\ \overline{2}2 & +-\end{matrix}, \qquad \begin{matrix}+0 & 0-\\ -0 & 0+\\ \overline{2}+ & +0\\ 2- & -0\end{matrix}, \qquad \begin{matrix}0+ & -0\\ 0- & +0\\ +\overline{2} & 0+\\ -2 & 0-\end{matrix},$$

$$\begin{matrix}00 & -+\\ +- & 00\\ -+ & 2\overline{2}\\ 00 & -+\end{matrix}, \qquad \begin{matrix}+- & 00\\ 00 & -+\\ 00 & -+\\ -+ & 2\overline{2}\end{matrix}, \qquad \begin{matrix}0+ & -0\\ 0- & +0\\ -0 & 2-\\ +0 & \overline{2}+\end{matrix}, \qquad \begin{matrix}+0 & 0-\\ -0 & 0+\\ 0- & -2\\ 0+ & +\overline{2}\end{matrix},$$

$$\begin{array}{ll} 2- & \overline{2}+ \\ -0 & +0 \\ -0 & +0, \\ 0+ & 0- \end{array} \qquad \begin{array}{ll} +\overline{2} & -2 \\ 0+ & 0- \\ 0+ & 0-, \\ -0 & +0 \end{array} \qquad \begin{array}{ll} +0 & -0 \\ \overline{2}+ & 2- \\ 0- & 0+, \\ +0 & -0 \end{array} \qquad \begin{array}{ll} 0+ & 0- \\ +\overline{2} & -2 \\ -0 & +0, \\ 0+ & 0- \end{array}$$

$$\begin{array}{ll} +0 & -0 \\ 0- & 0+ \\ \overline{2}+ & 2-, \\ +0 & -0 \end{array} \qquad \begin{array}{ll} 0+ & 0- \\ -0 & +0 \\ +\overline{2} & -2, \\ 0+ & 0- \end{array} \qquad \begin{array}{ll} 0+ & 0- \\ -0 & +0 \\ -0 & +0, \\ 2- & \overline{2}+ \end{array} \qquad \begin{array}{ll} +0 & -0 \\ 0- & 0+ \\ 0- & 0+, \\ -2 & +\overline{2} \end{array}$$

$$\begin{array}{ll} 2- & -0 \\ -0 & 0+ \\ -0 & 0+, \\ 0+ & +\overline{2} \end{array} \qquad \begin{array}{ll} +\overline{2} & 0+ \\ 0+ & -0 \\ 0+ & -0, \\ -0 & 2- \end{array} \qquad \begin{array}{ll} 0+ & -0 \\ +\overline{2} & 0+ \\ -0 & 2-, \\ 0+ & -0 \end{array} \qquad \begin{array}{ll} +0 & 0- \\ \overline{2}+ & +0 \\ 0- & -2, \\ +0 & 0- \end{array}$$

$$\begin{array}{ll} +0 & \overline{2}+ \\ 0- & +0 \\ 0- & +0, \\ -2 & 0- \end{array} \qquad \begin{array}{ll} 0+ & +\overline{2} \\ -0 & 0+ \\ -0 & 0+, \\ 2- & -0 \end{array} \qquad \begin{array}{ll} 0+ & -0 \\ -0 & 2- \\ +\overline{2} & 0+, \\ 0+ & -0 \end{array} \qquad \begin{array}{ll} +0 & 0- \\ 0- & -2 \\ \overline{2}+ & +0, \\ +0 & 0- \end{array}$$

References

[1] 4ti2 team, "4ti2—a software package for algebraic, geometric and combinatorial problems on linear spaces," available at http://www.4ti2.de.

[2] W. Bruns, B. Ichim, and C. Söger, "Normaliz. Algorithms for rational cones and affine monoids." available from http://www.math.uos.de/normaliz.

[3] D. Grayson and M. Stillman, "Macaulay2, a software system for research in algebraic geometry," available at http://www.math.uiuc.edu/Macaulay2.

[4] R. Hemmecke, A. Takemura, and R. Yoshida, "Computing holes in semi-groups and its applications to transportation problems," *Contributions to Discrete Mathematics*, vol. 4, no. 1, 2009.

[5] T. Kahle, J. Rauh, and S. Sullivant, "Positive margins and primary decomposition," *Journal of Commutative Algebra*, 2014, accepted.

[6] J. Rauh and S. Sullivant, "Lifting markov bases and higher codimension toric fiber products," *arXiv*, 2014. [Online]. Available: http://arxiv.org/abs/1404.6392